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Vertex operator realization and representations of hyperbolic Kac–Moody algebra $\hat{A}_1^{(1)}$

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Abstract. The main features of a hyperbolic Kac–Moody algebra (denoted by $\hat{A}_1^{(1)}$), which appears in the dimensional reduction of $N = 1$ supergravity from four to one dimensions, are presented. A vertex construction is exhibited and the structure of the fundamental representations is discussed. The vertex operator realization is presented in full generality, i.e. for any indefinite Kac–Moody algebras.

1. Introduction

In recent years the theory of (affine) Kac–Moody [1] algebras (KMAs) has attracted considerable attention from both the mathematical and physical points of view and it has now become a standard subject in modern Lie algebra textbooks (see, e.g., [2]). However little is known about infinite-dimensional KMAs defined by an indefinite Cartan matrix besides those in Kac's book [3], to which we will refer throughout this paper.

From time to time this type of Lie algebra has appeared in physics literature, mainly in the context of string theories. The possible relevance of infinite Lie algebras in the context of (super)gravity theories was pointed out by Julia [4] several years ago. Recently Nicolai [5] has shown that an hyperbolic extension of SL$(2, R)$ appears in the dimensional reduction of ($N = 1$) supergravity from four to one dimension.

This particular hyperbolic algebra, which we shall denote $\hat{A}_1^{(1)}$ and will discuss here, has already been considered in some detail by Feingold and Frenkel [6].

The aim of this paper is to present a vertex construction of the hyperbolic algebra, along the lines of the covariant vertex construction suggested by Goddard and Olive [7], and to discuss several features of the fundamental representations.

The paper is organized as follows. In section 2 we recall a few properties of hyperbolic KMAs and give the Cartan matrix, the Dynkin diagram and the structure of the roots of $\hat{A}_1^{(1)}$. In section 3 we present the vertex construction of the algebra and in section 4 we discuss some properties of the fundamental representation. At the end we present some conclusions and we point out some of the many, as yet, unsolved problems.

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We mainly focus our attention on the particular case of algebra $\hat{A}_1^{(1)}$, but we try to keep our presentation as general as possible in order to exhibit features which are common to all Lorentzian KMAs, which will be discussed in more detail elsewhere [8].

2. The hyperbolic KMA $\hat{A}_1^{(1)}$

Let us recall a few definitions which we need to define a KMA [3, 2].

A $(d \times d)$ matrix $A = [a_{ij}]$ is called a generalized Cartan matrix (GCM) if it satisfies the following conditions:

(i) $a_{ij} \in Z$;

(ii) $a_{ii} = 2$;

(iii) $a_{ij} \leqslant 0$ $(i \neq j)$;

(iv) $a_{ij} = 0$ implies $a_{ji} = 0$.

A matrix A is described as indecomposable if it cannot be reduced to a block diagonal form by shuffling rows and columns and symmetrizable if the matrix DA is symmetric, D being an invertible diagonal matrix.

We associate with a GCM a Dynkin diagram (DD), denoted sometimes $S(A)$, with the following properties:

(a) $S(A)$ has d vertices;

(b) if $a_{ij}a_{ji} = n \leqslant 4$, the vertices i and j are joined by $|a_{ij}| \geqslant |a_{ji}|$ lines;

(c) if $|a_{ij}| \geqslant |a_{ji}|$ $(|a_{ij}| \leqslant |a_{ji}|)$, we put on the lines (ij) an arrow pointing, respectively, towards the vertex j (i);

(d) if $n > 4$ the vertices i and j are connected by a boldfaced line on which an ordered pair of integers, $|a_{ij}|$ and $|a_{ji}|$, is written. This case will appear only for $d = 2$ for hyperbolic (Hyp) KMAs (defined later). Note that sometimes in the literature when condition (b) is satisfied the vertices are joined by n lines. Clearly A is indecomposable if and only if the corresponding $S(A)$ is a connected diagram.

We associate with a given GCM A a complex Lie algebra defined by $3d$ generators, E_i, F_i and H_i which satisfy the following commutation and Serre relations

$$[E_i, F_j] = a_{ij} H_i \tag{1}$$

$$[H_i, H_j] = 0 \tag{2}$$

$$[H_i, E_j] = a_{ij} E_j \tag{3}$$

$$[H_i, F_j] = -a_{ij} F_j \tag{4}$$

$$(adE_i)^{1-a_{ij}} E_j = (adF_i)^{1-a_{ij}} F_j = 0 \qquad (i \neq j). \tag{5}$$

The algebra can be written in the following form (triangular decomposition)

$$G(A) = N_- \oplus H \oplus N_+$$

where H is the Cartan subalgebra and $N_-(N_+)$ are, respectively, the linear span of $F_i(E_i)$.

We can have three cases:

(i) $\det A > 0$ corresponding to finite KMAs;

(ii) $\det A = 0$, with rank of A equal to $d - 1$ and determinant of any leading principal submatrix positive, corresponding to Aff KMAs; and

(iii) $\det A \leqslant 0$ corresponding to indefinite KMAs.

The hyperbolic KMAs are a particular case of the indefinite KMAs with the further condition that every leading submatrix decomposes into constituents of finite and/or affine type or, in an equivalent way, by deleting a vertex of the corresponding $S(A)$ one obtains DD of finite or affine KMAs.

We can make the following distinctions [9]:

• strictly hyperbolic (SHyp) if every leading principal submatrix decomposes into constituents of finite type;

• purely hyperbolic (PHyp) if every leading principal submatrix decomposes into constituents of affine type;

• hyperfinite (HypF) if at least one leading principal submatrix decomposes into constituents of finite type;

• hyperaffine (HypA) if at least one leading principal submatrix is of affine type.

A symmetrizable GCM A and the corresponding KMA is said to be Lorentzian if the matrix A has signature $(d-1,1)$.

Clearly SHyp and PHyp are Hyp and every Hyp matrix is either HypF or HypA or both.

One can show [10, 9]:

Theorem 1. All the GCM of type Hyp (Shyp, Phyp, HypF and HypA) are Lorentzian.

A classification of hyperbolic algebras has been made in [11, 12] generalizing previous results obtained in [3]. In [11] all the DDs have been drawn, resulting in 238 DDs (of rank $\geqslant 3$) of which 85 are SHyp and 142 DDs correspond to symmetric or symmetrizable GCM. The highest rank of the Hyp algebras is 10 and E_{10} belongs to this class which has appeared several time in the context of string theories.

For Hyp algebras corresponding to symmetrizable GCM, a set of simple roots (SR) α_i can be introduced in such way that the matrix A determines a symmetric bilinear form $(\cdot,\cdot)$ in the root space (H^*) defined by

$$a_{ij} = (\alpha_i, \alpha_j) = \alpha_i \cdot \alpha_j . \tag{6}$$

An affine KMA can be obtained by adding to the lattice Γ of roots of a finite Lie algebra G (horizontal algebra) a light-like vector K^+:

$$(K^+, K^+) = 0 \qquad (K^+, \alpha_i) = 0. \tag{7}$$

The affine root is obtained by adding K^+ to the lowest root of G (affinization procedure).

A class of HypA is obtained by adding to the lattice of roots of affine KMA another light-like root K^- such that

$$(K^-, \alpha_i) = (K^-, K^-) = 0 \qquad (K^-, K^+) = 1. \tag{8}$$

Double-affinization is called the procedure of adding to an Aff KMA an SR containing K^- such that this new root has a scalar product equal to -1 with the affine root and zero with the other SRs.

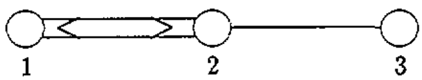
A classification of double-affinized KMAs has been obtained by Ogg [13]. In the original paper the procedure was called superaffinization but we prefer to change the name to double-affinization to avoid confusion with the procedure of affinization of superalgebras.

Let us remark that in the case of Aff KMAs K^+ is a root (not simple), while in the case of Hyp KMAs K^- is not always a root.

The symmetric GCM defining the $\hat{A}_1^{(1)}$ algebra is [6, 3]

$$A = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

and the corresponding $S(A)$ is



The simple roots are, denoting by α the root of A_1, $(\alpha_i, \alpha_i) = 2$

$$\begin{aligned} \alpha_1 &= \alpha \\ \alpha_2 &= -\alpha + K^+ \\ \alpha_3 &= -K^+ - K^-. \end{aligned} \tag{9}$$

The algebra is defined by equations (1) to (5), where $E_i(F_i)$ $(i = 1, 2, 3)$ corresponds, respectively, to $\alpha_i(-\alpha_i)$ and a_{ij} is given in terms of the SR by equation (6). We recall that a graded Lie algebra $G = \oplus_{-\infty}^{\infty} G_i$, generated by $G_0 \oplus G_1 \oplus G_{-1}$, simple (i.e. not containing non-trivial homogeneous ideals) is said to be of finite growth [14] if the dimension of the space G_i grows as a power of $|i|$. From Kac's theorem in [14] we see that the algebra $\hat{A}_1^{(1)}$ is not of finite growth.

Clearly it is both HypF and HypA; in fact, deleting the vertex corresponding to the SR $\alpha_1, \alpha_2, \alpha_3$ one finds the subalgebras $A_2, A_1 + A_1, A_1^{(1)}$, respectively.

The set of roots r $(\Delta = \{r\})$ [6] is given by $r = \sum_{i=1}^{3} k_i \alpha_i$ where the triple of integers is constrained by

$$(k_1^2 + k_2^2 + k_3^2) - 2k_1k_2 - k_2k_3 \leqslant 1. \tag{10}$$

In the following we will specify a root by a triple of integers $[k_1, k_2, k_3]$, denoting its length by a superscript on the triple and we will use the following notation:

$$\Delta_+ = \{r = [k_1, k_2, k_3]\, k_i \in Z_+\} \tag{11}$$

$$\Delta^L = \{r = [k_1, k_2, k_3]^L\, r \cdot r = L,\ L \in 2Z,\ k_i \in Z\}. \tag{12}$$

In particular we have

$$\Delta_+^2 = \{[\pm\sqrt{1 + k_3 n} + k_3 + n, k_3 + n, k_3]\, n, \sqrt{1 + k_3 n}, k_3 \in Z_+, k_3 \neq 0\} \tag{13}$$

$$\Delta_+^0 = \{[\pm\sqrt{k_3 n} + k_3 + n, k_3 + n, k_3]\, n, \sqrt{k_3 n}, k_3 \in Z_+, k_3 \neq 0\} \tag{14}$$

$$\Delta_+^{-2l} = \{[\pm\sqrt{nk_3 - l} + k_3 + n, k_3 + n, k_3]\, l \geqslant 0, \sqrt{nk_3 - l}, n, k_3 \in Z_+\}. \tag{15}$$

It is useful to have a formula for computing the triple of (finite) numbers $[k_1, k_2, k_3]$ as a function of the height (ht) $(ht = k_1 + k_2 + k_3)$ and of the length (L) of the roots.

We have

$$k_1 = \frac{4ht - 5k_3 \pm \sqrt{k_3(8ht - 23k_3) + 8L}}{8} \in Z_+. \tag{16}$$

The expression in the square root is positive if

$$k_3 \in [X_+, X_-] \tag{17}$$

$$X_\pm = (4ht \pm \Delta)/23 \qquad \Delta = 4\sqrt{ht^2 + \tfrac{23}{2}L}. \tag{18}$$

$\Delta \geqslant 0$ implies that roots exist if

$$ht \leqslant \sqrt{\tfrac{23}{2}|L|}. \tag{19}$$

So for any fixed ht the set of roots can be obtained by the following algorithm

(i) Consider $L \in Z_+$ with $L \geqslant -2ht/23$.

(ii) Compute the corresponding $X_\pm$ and then the allowed values of $k_3 \in Z_+$.

(iii) For any integer value of k_3 compute k_1 and verify it is an integer.

We remark that

(a) the only roots of type $[k_1, k_2, 0]$ are $(ht = k_1 + k_2)$

$$[ht - n, n, 0]^2 \qquad n = (ht \pm 1)/2 \in Z_+;$$

$$[ht/2, ht/2, 0]^0 \qquad ht \in 2Z_+$$

(b) the only roots of type $[0, k_2, k_3]$ are (all of length = 2)

$$[0,1,0] \qquad [0,0,1] \qquad [0,1,1]$$

(c) there are no roots of the type $[k_1, 0, k_3]$;

(d) the roots of negative length always have $k_3 \neq 0$.

The roots of low height (ht) are

$$ht = 1\ \{[1,0,0]^2; [0,1,0]^2; [0,0,1]^2\}$$

$$ht = 2\ \{[1,1,0]^0; [0,1,1]^2\}$$

$$ht = 3\ \{[2,1,0]^2; [1,2,0]^2; [1,2,1]^0\}$$

$$ht = 4\ \{[2,2,0]^0; [2,1,1]^2; [1,2,1]^0\}$$

$$ht = 5\ \{[3,2,0]^2; [2,3,0]^2; [2,2,1]^{-2}; [1,2,2]^2\}$$

$$ht = 6\ \{[3,3,0]^0; [3,2,1]^0; [2,2,2]^0; [2,3,1]^{-2}\}.$$

Note that the first negative length root appears at $ht = 5$. The multiplicity $(m(r))$ of roots does not only depend on length, as remarked in [6, 10] and it can be computed by means of Peterson's recurrent formula [3].

Let $G(A)$ be a KMA with symmetrizable GCM; set

$$C_r = \sum_{n\geqslant 1} n^{-1} m(r/n) \tag{20}$$

where r belongs to Γ^+ ($\Gamma^+ = \sum n_i\alpha_i,: n_i \in Z_+$), ρ is the 'unit' root

$$(\rho, r_i) = 1 \qquad \forall i \leqslant d. \tag{21}$$

For $\hat{A}_1^{(1)}$ the unit root is $\rho = [-9/2, -5, -2]$. Then the recurrent formula, which is proven in [9], reads

$$(r, r-2\rho)C_r = \sum_{r'+r''=r} (r', r'')C_{r'}C_{r''}. \tag{22}$$

Finally we recall several definitions and properties of the Weyl group for KMAs, which we shall use in section 4.

We recall [3] that the Weyl group (W) of a KMA is a discrete group of isometries of the dual of the Cartan subalgebra generated by the reflections with respect to the SRs (fundamental reflections). The elements of W which are obtained as a product of an even number of fundamental reflections form a normal subgroup Wo of W, called either a conformal Weyl group or an even subgroup. Clearly, for Hyp KMAs, Wo $\in \mathrm{SO}^+(d-1,1)$, while W $\in \mathrm{O}^+(d-1,1)$.

In the case of $\hat{A}_1^{(1)}$ the Weyl group is generated by the three fundamental reflections whose action on the SRs is given by

$$w_{\alpha_i}(\alpha_j) = \alpha_j - a_{ij}\alpha_i. \tag{23}$$

This group is discussed in detail in [6]. It can be identified with the extended modular group ($\mathrm{PGL}_2(\mathrm{Z})$) while Wo is isomorphic to the modular group ($\mathrm{PSL}_2(\mathrm{Z})$). We remind ourselves that a root r is called real if there exists an element $w \in$ W such that $w(r)$ is an SR. Let us recall several important properties of the Weyl group:

(i) the bilinear form $(\cdot,\cdot)$, the sets Δ and Δ^{im} are invariant under the action of W;

(ii) the dimension of a space of a root is equal to the dimension of the space of the reflected root;

(iii) contrary to what happens in the case of finite Lie algebras, fixed points may exist in the root space of a KMA, e.g. the root $\alpha_1 + \alpha_2 = K^+ \in A_1^{(1)}$ is fixed for reflections with respect to α_1 and α_2. In fact K^+ is orthogonal to both α_1 and α_2. It is easy to show that there are no fixed points for $\hat{A}_1^{(1)}$.

Considering the set of roots

$$R_+ = \{r \in \Delta_+^{\mathrm{im}} : (r, \alpha_i) \geqslant 0\ \forall i\} \tag{24}$$

we can obtain all the other imaginary roots, in fact

$$\Delta_+^{\mathrm{im}} = \cup_{w\in \mathrm{W}} w(R_+).$$

In the case of $\hat{A}_1^{(1)}$ the roots $r \in R_+$ must satisfy the conditions

$$\frac{2ht - 3k_3}{3} \geqslant k_1 \geqslant \frac{ht - k_3}{2} \qquad \forall ht \geqslant \frac{13}{2}k_3, k_3, k_1 \in Z_+ \tag{25}$$

$$\frac{ht + k_3}{3} \geqslant k_1 \geqslant \frac{ht - k_3}{2} \qquad \forall \frac{13}{2}k_3 > ht \geqslant 5k_3, k_3, k_1 \in Z_+. \tag{26}$$

A complete classification of roots in R_+ can be given in terms of the set $\{ht, L, k_3\}$:

(i) for $k_3 = 0$, $r = [ht/2, ht/2, 0]^0$ $\forall ht \in 2Z$;

(ii) for $ht = 5k_3$, $r = [2k_3, 2k_3, k_3]^{-2k_3}$;

(iii) for $ht = 6k_3 + n$, $r = [k_1, ht - k_1 - k_3, k_3]^L$ with $(9k_3 + 2n)/4 \leqslant k_1 \leqslant (5k_3 + n)/2$ $k_1 \in Z$ and $L = k_1^2 + k_3^2 + (ht - k_1 - k_3)(ht - 3k_1 - 2k_3)$.

A computer-calculated table of multiplicities of roots of low ht, belonging to R_+, for $\hat{A}_1^{(1)}$, is reported in [3], p 149.

The roots which are not real are called imaginary. Clearly

$$\Delta = \Delta^{\mathrm{re}} \cup \Delta^{\mathrm{im}} \tag{27}$$

and we have

$$\Delta^{\mathrm{re}} = \{r \in \Delta : (r, r) = 2\} \tag{28}$$

$$\Delta^{\mathrm{im}} = \{r \in \Delta : (r, r) \leqslant 0\}. \tag{29}$$

3. Vertex operator construction

The well known vertex construction of an Aff KMA can be generalized to the case of Ind KMAs along the lines of the covariant construction of Goddard and Olive [7].

Let us introduce d Fubini–Veneziano fields ($\mu = 1, \ldots, d$)

$$Q^\mu(z) = q^\mu - \mathrm{i}p^\mu \ln z + \mathrm{i}\sum_{n \neq 0} \frac{\alpha_n^\mu}{n} z^{-n} \tag{30}$$

where

$$[q^\mu, p^\nu] = \mathrm{i}g^{\mu\nu} \qquad \mathrm{sign}(g) = (-, +, +, \cdots, +) \tag{31}$$

$$[\alpha_n^\mu, \alpha_m^\nu] = n\delta_{n+m,0}g^{\mu\nu} \qquad \alpha_m^{\mu\dagger} = \alpha_{-m}^\mu. \tag{32}$$

These fields are defined on a d-dim Minkowskian torus and satisfy periodical boundary conditions on the roots lattice Γ.

Let us define

$$U^r(z) = : \mathrm{e}^{\mathrm{i}r \cdot Q(z)} : \tag{33}$$

where : : denotes normal ordered product and

$$r \cdot Q^{(n)}(z) = \frac{\mathrm{i}d^n}{n!\,\mathrm{d}z^n} r \cdot Q(z). \tag{34}$$

$U^r(z)$ are the vertex operators (VO) which are introduced for the vertex realization of an Aff KMA. It is possible to introduce a generalized VO (GVO) (see [15, 16]) by means of the following ordered product

$$:r_1 \cdot Q^{(n_1)}(z) r_2 \cdot Q^{(n_2)} \dots r_N \cdot Q^{(n_N)}(z) U^r(z): \tag{35}$$

where $r_i \in \Gamma$ and $n_i \in Z_+$. It is convenient to express a GVO in a different basis [17] introducing a set of Schur polynomials, which are defined by the following formal expansion

$$\exp\left(\sum_{m>0} \frac{c_m}{m} z^{-m}\right) = \sum_n P_n(c) z^{-n} \tag{36}$$

where c_m are commuting variables. So we have

$$P_n(c) = \sum_{k_l} \frac{1}{k_l!}\left(\frac{c_l}{l}\right)^{k_l} \qquad \sum_l l k_l = n. \tag{37}$$

The new basis will be formed by fields which are represented as Schur polynomials in the fields $r \cdot Q^{(l)}(z)$ $(1 \leqslant l \leqslant n)$:

$$\begin{aligned} P_n(r \cdot Q^{(l)}(z)) &= :\left(\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}z^n} U^r(z)\right) U^{-r}(z): \\ &= \lim_{z_1 \to z} \frac{1}{n!} : \frac{\partial^n}{\partial z_1^n} U^r(z_1) U^{-r}(z):. \end{aligned} \tag{38}$$

This equation has to be read as a Schur polynomial, the variable c_l being now replaced by the field $r \cdot Q^{(l)}$. It follows that a GVO is an ordered product of Schur polynomials and standard VO:

$$U^{\{r,r_i\}}_{\{(n_i)\}}(z) = :\prod_i P_{n_i}(r_i \cdot Q^{(l_i)}(z)) U^r(z): \tag{39}$$

which can be explicitly written in the following form:

$$\begin{aligned} U^{\{r,r_i\}}_{\{(n_i)\}}(z) &= :\prod_i \lim_{z_i \to z} \frac{1}{n_i!} \frac{\partial^{n_i}}{\partial z_i^{n_i}} U^{r_i}(z_i) U^{-r_i}(z) U^r(z): \\ &= :\prod_i U^{r_i (n_i)}(z) U^{-r_i}(z) U^r(z):. \end{aligned} \tag{40}$$

We can make a Laurent expansion of a GVO

$$U^{\{r,r_i\}}_{\{(n_i)\}} = \sum_m A^{\{r,r_i\}}_{m\{(n_i)\}} z^{-m-1}. \tag{41}$$

The terms in equation (41) with $m = 0$ are

$$A^{\{r,r_1,\dots,r_N\}}_{\{(n_1),\dots,(n_N)\}} = \frac{1}{2\pi \mathrm{i}} \oint_{C_0} \mathrm{d}z\, U^{\{r,r_1,\dots,r_N\}}_{\{(n_1),\dots,(n_N)\}}(z) \tag{42}$$

where the integral is performed along a closed path C_0, including the point $z = 0$.

With any root of length $L = r^2$ we associate a GVO such that

$$\tfrac{1}{2}r^2 + \sum_i n_i = 1 \tag{43}$$

so for $L = 2$ the corresponding GVO is the standard VO while for $L = 0$ it is the photonic VO [7].

Let us point out that equation (43) is connected with the conformal symmetry of the GVOs, which will be discussed elsewhere [8], and that not all the sets (n_i), which satisfy equation (43), in fact appear in the construction of the algebra.

Now we have the following proposition.

Proposition 1. The product of two GVOs can be written in the following form, for $|z| > |\xi|$

$$U^{\{r,r_i\}}_{\{(n_i)\}}(z)U^{\{s,s_j\}}_{\{(n_j)\}}(\xi) = \sum_{k_i,k_j}^{n_i,n_j} \chi^{\{r,s,r_i,s_j\}}_{\{(k_i),(k_j)\}} : \prod_{i,j}[U^{r_i(n_i-k_i)}(z)U^{-r_i}(z)$$
$$\times U^{s_j(n_j-k_j)}(\xi)U^{-s_j}(\xi)U^{r}(z)U^{s}(\xi)] : (z-\xi)^{r\cdot s-\sum_i k_i-\sum_j k_j} \tag{44}$$

where

$$\chi^{\{r,s,r_i,s_j\}}_{\{(k_i),(k_j)\}} = \prod_{i,j} \lim_{\substack{z_i\to z\\ \xi_i\to\xi}} \frac{\partial^{k_i+k_j}}{k_i!k_j!\partial z_i^{k_i}\partial\xi_j^{k_j}}[(z_i-\xi)^{r_i\cdot(s-s_j)}(z-\xi_j)^{(r-r_i)\cdot s_j}(z_i-\xi_j)^{r_i\cdot s_j}]$$
$$\times [(z-\xi)^{r_i\cdot r_j+r_i\cdot(s-s_j)+(r-r_i)\cdot s_j-\sum k_i-\sum k_j}]^{-1} \tag{45}$$

and

$$U^{r_i(n_i)}(z) = \frac{1}{n_i!}\frac{\mathrm{d}^{n_i}}{\mathrm{d}z^{n_i}}U^{r_i}(z). \tag{46}$$

Proof. From the well known relation [18]

$$U^{r}(z)U^{s}(\xi) =: U^{r}(z)U^{s}(\xi) : (z-\xi)^{r\cdot s} \qquad \text{for } |\xi| < |z| \tag{47}$$

it is possible to write the product of two GVOs as an operator product expansion (OPE)

$$U^{\{r,r_i\}}_{\{(n_i)\}}(z)U^{\{s,s_j\}}_{\{(n_j)\}} = \prod_{ij} \lim_{z_i\to z}\lim_{\xi_j\to\xi} : \frac{1}{n_i!n_j!}\frac{\partial^{n_i+n_j}}{\partial z_i^{n_i}\partial\xi_j^{n_j}}[U^{r_i}(z_i)U^{-r_i}(z)$$
$$\times U^{r}(z)U^{s_j}(\xi_j)U^{-s_j}(\xi)U^{s}(\xi)] : (z_i-\xi)^{r_i\cdot(s-s_j)}(z-\xi_j)^{(r-r_i)\cdot s_j}$$
$$\times (z_i-\xi_j)^{r_i\cdot s_j}(z-\xi)^{r\cdot s-r_i\cdot s-r\cdot s_j+r_i\cdot s_j}. \tag{48}$$

By applying the Leibnitz rule for derivation this equation can be written as

$$U^{\{r,r_i\}}_{\{(n_i)\}}(z)U^{\{s,s_j\}}_{\{(n_j)\}} = \sum_{k_ik_j}^{n_i n_j}\prod_{ij} \lim_{z_i\to z}\lim_{\xi_j\to\xi} : \frac{1}{n_i!n_j!}\binom{n_i}{k_i}\binom{n_j}{k_j}U^{r_i(n_i-k_i)}(z_i)$$
$$\times U^{-r_i}(z)U^{r}(z)U^{s_j(n_j-k_j)}(\xi_j)U^{-s_j}(\xi)U^{s}(\xi) : \frac{\partial^{n_i+n_j}}{\partial z_i^{n_i}\partial\xi_j^{n_j}}$$
$$\times[(z_i-\xi)^{r_i\cdot(s-s_j)}(z-\xi_j)^{(r-r_i)\cdot s_j}(z_i-\xi_j)^{r_i\cdot s_j}(z-\xi)^{r\cdot s-r_i\cdot s-r\cdot s_j+r_i\cdot s_j}] \tag{49}$$

taking the limit we get equation (44). □

Let us warn that the label k_i in equation (44) and in the following ones should not be confused with one used in section 2 to express a root in terms of SRs.

To compute the commutation relation of two GVOs we note that, by changing the two integration variables z and ξ, we have

$$\chi^{\{s,r,s_j,r_i\}}_{\{(k_j),(k_i)\}} = (-1)^{-\sum_i k_i - \sum_j k_j} \chi^{\{r,s,r_i,s_j\}}_{\{(k_i),(k_j)\}}. \tag{50}$$

From equations (50) and (44) it follows that

$$U^{\{s,s_j\}}_{\{(n_j)\}}(\xi) U^{\{r,r_i\}}_{\{(n_i)\}}(z) = (-1)^{r\cdot s} U^{\{r,r_i\}}_{\{(n_i)\}}(z) U^{\{s,s_j\}}_{\{(n_j)\}}(\xi) \qquad \text{with} \qquad |z| \leqslant |\xi|. \tag{51}$$

Finally, in complete analogy with the case of Aff KMAs, we can write

$$\begin{aligned} A^{\{r,r_i\}}_{\{(n_i)\}} A^{\{s,s_j\}}_{\{(n_j)\}} &- (-1)^{r\cdot s} A^{\{s,s_j\}}_{\{(n_j)\}} A^{\{r,r_i\}}_{\{(n_i)\}} \\ &= \frac{1}{(2\pi \mathrm{i})^2} \left[\oint \mathrm{d}z \oint_{|\xi|>|z|} \mathrm{d}\xi - \oint \mathrm{d}z \oint_{|z|<|\xi|} \mathrm{d}\xi \right] U^{\{r,r_i\}}_{\{(n_i)\}}(z) U^{\{s,s_j\}}_{\{(n_j)\}}(\xi) \\ &= \frac{1}{(2\pi \mathrm{i})^2} \oint_0 \mathrm{d}z \oint_z \mathrm{d}\xi \, U^{\{r,r_i\}}_{\{(n_i)\}}(z) U^{\{s,s_j\}}_{\{(n_j)\}}(\xi). \end{aligned} \tag{52}$$

In order for the left-hand side of equation (52) to be a commutator, we have to introduce the cocycles, which have been constructed in [18].

To compute the right-hand side of equation (52) we have to evaluate the residues at poles of order

$$p = \sum k_i + \sum k_j - r \cdot s \qquad \text{for } z = \xi \tag{53}$$

in the OPE.

From equation (43) for roots r and s it follows that the right-hand side of equation (52) has no poles for

$$(r+s)^2 \geqslant 4 \tag{54}$$

so we obtain a condition on the length of roots which ensures the absence of poles.

When equation (54) is satisfied the right-hand side of equation (52) vanishes.

Let us remark that not all the operators are relevant for the construction of the algebra as there is a class of GVOs which vanishes. In fact if $U^{\{r,r_i\}}_{\{(n_i)\}}(z)$ can be written as a total z derivative of a GVO, then $A^{\{r,r_i\}}_{\{(n_i)\}} = 0$. This property is also a consequence of the conformal symmetry.

We can summarize the relevant commutation relations in the following formula, where suitable cocycles are supposed to have been included,

$$\left[A^{\{r,r_i\}}_{\{(n_i)\}}, A^{\{s,s_j\}}_{\{(n_j)\}} \right] = \frac{1}{(2\pi \mathrm{i})^2} \oint_0 \mathrm{d}z \oint_z \mathrm{d}\xi \, U^{\{r,r_i\}}_{\{(n_i)\}}(z) U^{\{s,s_j\}}_{\{(n_j)\}}(\xi). \tag{55}$$

Equation (55) can be written, after some calculations using our proposition,

$$\begin{aligned} \left[A^{\{r,r_i\}}_{\{(n_i)\}}, A^{\{s,s_j\}}_{\{(n_j)\}} \right] &= \sum_{k_i,k_j}^{n_i,n_j} \frac{\chi^{\{r,s,r_i,s_j\}}_{\{(k_i),(k_j)\}}}{2^{-r\cdot s + \sum k_i + \sum k_j} (-r\cdot s + \sum k_i + \sum k_j - 1)!} \\ &\quad \times \prod_{b=0}^{2N+2M} \sum_{l_b}^{x(b)} (-1)^{l_b} \binom{x(b)}{l_b} \\ &\quad \times A^{\{r+s, r_i, -r_i, s_j, -s_j, r, s\}}_{\{(l_i+n_i-k_i),(l_{N+i}),(l_{2N+j}+m_j-k_j),(l_{2N+2M+j}),(y),(l_0)\}} \end{aligned} \tag{56}$$

where $1 \leqslant i \leqslant N$, $1 \leqslant j \leqslant M$, $x(b) = -r \cdot s + \sum k_i + \sum k_j - 1 - \sum_{c<b} l_c$ and $y = -r \cdot s + \sum k_i + \sum k_j - 1 - \sum_{d>0} l_d$.

Moreover in the A operator the labels have been ordered in such a way that the Kth $(1 \leqslant K \leqslant 6)$ set of subscript labels denotes the order of derivation of the set of vos specified by the $(K+1)$th set of superscript labels.

In particular if $s = -r$ and $s_i = -r_i$ we get

$$\left[A^{\{r,r_1,\ldots,r_N\}}_{\{(n_1),\ldots(n_N)\}}, A^{\{-r,-r_1,\ldots,-r_N\}}_{\{(n_1),\ldots,(n_N)\}}\right] = \chi^{\{r,-r,r_1,-r_1,\ldots,r_N,-r_N\}}_{\{(n_1),(n_1),\ldots(n_N),(n_N)\}} r \cdot p. \quad (57)$$

From equation (57) we see the Cartan generators H_i are given by $\alpha_i \cdot p$. Note that for $n_i = 0$, for $r_i = 0$ and $r^2 = 2$ the χ coefficient in the right-hand side is equal to 1.

The χ coefficients can be explicitly computed as a combination of factorials

$$\chi^{\{r,s,r_i,s_j\}}_{\{(k_i),(k_j)\}} = \prod_i^N \prod_j^M \sum_{l_i=0}^{k_i} \sum_{l_j=0}^{k_j} (-1)^{l_j} \binom{s_j \cdot (r - r_i)}{k_j - l_j} \binom{r_i \cdot s_j}{l_i}$$
$$\times \binom{r_i \cdot (s - s_j)}{k_i - l_i} \binom{r_i \cdot s_j - l_i}{l_j}. \quad (58)$$

We end this section with a few relevant remarks:

(i) no central charge is present in the case of $\hat{A}_1^{(1)}$;

(ii) the 'derivation' D of the Aff subaglebra $A_1^{(1)}$, which can be written as $D = -K^- \cdot p$ and does not belong to $A_1^{(1)}$, belongs to $\hat{A}_1^{(1)}$;

(iii) the complexity of equation (56) may suggest that our choice is not the most convenient one. However the right-hand side of this equation can be easily implemented in a computer algorithm to perform numerical calculations.

Let us explicitly write a few relevant commutation relations for the simple roots of $\hat{A}_1^{(1)}$.

We have $(i = 1, 2, 3)$

$$[A^{\alpha_i}, A^{-\alpha_i}] = \alpha_i \cdot p \quad (59)$$

with

$$[\alpha_i \cdot p, A^{\pm\alpha_j}] = \pm(\alpha_i \cdot \alpha_j) A^{\pm\alpha_j} = \pm a_{ij} A^{\pm\alpha_j} \quad (60)$$

where use has been made of equations (42), (33) and (31).

Then we have

$$[A^{\alpha_1}, A^{\alpha_3}] = 0 \qquad (a_{13} = 0) \quad (61)$$

$$[A^{\alpha_2}, A^{\alpha_3}] = A^{\alpha_2+\alpha_3} \qquad (a_{23} = -1) \quad (62)$$

$$[A^{\alpha_2}, [A^{\alpha_2}, A^{\alpha_3}]] = 0 \quad (63)$$

as $(\alpha_2 \cdot (\alpha_2 + \alpha_3) = 1)$

$$[A^{\alpha_1}, A^{\alpha_2}] = \frac{\chi^{\{\alpha_1,\alpha_2\}}}{2\pi i} \oint_{C_0} dz : \frac{1}{2}\left(\frac{d}{dz} - \frac{d}{d\xi}\right) U^{\alpha_1}(z) U^{\alpha_2}(\xi) : |_{z=\xi}$$
$$= \tfrac{1}{2} A^{\{\alpha_1+\alpha_2,\alpha_1-\alpha_2\}}_{(1)} \qquad (a_{12} = -2) \quad (64)$$

$$[A^{\alpha_1}, [A^{\alpha_1}, A^{\alpha_2}]] = \left[A^{\alpha_1}, \tfrac{1}{2} A^{\{\alpha_1+\alpha_2,\alpha_1-\alpha_2\}}_{(1)}\right] = \tfrac{1}{2}(\alpha_1 \cdot (\alpha_2 - \alpha_1)) A^{2\alpha_1+\alpha_2}. \quad (65)$$

Let us remark that this commutator is not vanishing in spite of the vanishing of the scalar product $\alpha_1\cdot(\alpha_1+\alpha_2)=0$ as the VO corresponding to the root $\alpha_1+\alpha_2$ is a GVO and then in the commutator a pole of order $\alpha_1\cdot(\alpha_1+\alpha_2)-1=-1$ appears, see equation (44). The commutator of A^{α_1} with $A^{2\alpha_1+\alpha_2}$ vanishes as $\alpha_1\cdot(2\alpha_1+\alpha_2)=2$.

So we have explicitly shown that the GVO construction is a realization of the algebra $\hat{A}_1^{(1)}$ in the Cartan–Chevalley basis, see equations (1)–(5).

4. Fundamental representations of $\hat{A}_1^{(1)}$

Many of the general properties of the representation theory of semi-simple Lie algebras hold for infinite KMAs. We refer to [3, 2] for a more detailed account and, for completeness, we only recall here a few properties that we need.

A representation of a KMA is called an integrable highest weight (HW) representation if the carrier space V is a direct (in general infinite) sum of finite-dimensional weight subspace. The elements of a weight subspace are eigenvectors of the operators $\Phi(H_i)$, $H_i\in H$ with eigenvalues $\lambda(H_i)$ and the dimension of the subspace V_λ is called the multiplicity ($m(\lambda)$) of weight λ.

There does exist one weight Λ, HW, with $m(\Lambda)=1$, which is annihilated by all operators $\Phi(E_i)$. For any simple root α_i there exist non-negative integers, n and n', such that for any $\phi\in V$

$$\Phi(E_i)^n\phi=\Phi(F_i)^{n'}\phi=0. \tag{66}$$

Any weight can be written as

$$\lambda=\Lambda-\sum_i k_i\alpha_i \qquad k_i\in Z_+. \tag{67}$$

The HW Λ, such that

$$\frac{2(\Lambda,\alpha_i)}{(\alpha_i,\alpha_i)}\in Z_+ \tag{68}$$

defines an irreducible HW representation (IR) of a KMA.

The $m(\lambda)$ can be computed by means of the Kac formula [2], which is proven in the appendix,

$$\sum_{w\in W}\epsilon(w)m(w(\Lambda+\rho)-(\lambda+\rho))=0 \tag{69}$$

where $\epsilon(w)=(-1)^{l(w)}$ is the parity of the Weyl element w, $l(w)$ being the lowest possible number of fundamental reflection needed to build w. $m(\lambda)$ is equal to $m(w(\lambda))$ $\forall w\in$ W.

The fundamental representations are specified by Λ_i such that

$$\frac{2(\Lambda_i,\alpha_j)}{(\alpha_j,\alpha_j)}=\delta_{ij}. \tag{70}$$

In the case of $\hat{A}_1^{(1)}$ we indicate a weight λ by the triple of integers (m_1, m_2, m_3).

$$\frac{2(\lambda,\alpha_i)}{(\alpha_i,\alpha_i)} = m_i. \tag{71}$$

In this case there exist three fundamental representations:

$$(1,0,0) \qquad \Lambda_1 = \frac{\alpha}{2} - K^+ + K^- \tag{72}$$

$$(0,1,0) \qquad \Lambda_2 = -K^+ + K^- \tag{73}$$

$$(0,0,1) \qquad \Lambda_3 = -K^+. \tag{74}$$

Starting from the HW, all the vectors of V are obtained by action of the operators $\Phi(F_i)$, as in the case of the simple Lie algebras. In the case of $\hat{A}_1^{(1)}$ the action of $\Phi(F_1)$ and $\Phi(F_2)$ spans an IR of the affine subalgebra $A_1^{(1)}$, starting from a HW for $A_1^{(1)}$, while $\Phi(F_2)$ and $\Phi(F_3)$ span an IR of the finite subalgebra A_2. The operator $\Phi(F_3)$ connects states belonging to IRs of $A_1^{(1)}$ at different levels, hence acting as a level-raising operator. So an IR of $\hat{A}_1^{(1)}$ looks like an infinite sum of representations of an increasing level of $A_1^{(1)}$.

The three fundamental IRs of $\hat{A}_1^{(1)}$, at the first 'level', reduce, respectively, to the level 1 spinorial IR, the basic IR and the trivial (one-dimensional) IR of $A_1^{(1)}$. For the representations of $A_1^{(1)}$ see [19, 20].

In figures 1, 2 and 3 we report a few states of the first three levels of the fundamental IRs of $\hat{A}_1^{(1)}$. In the figures the eigenvalues of $\Phi(H_i)$ and the multiplicity of the states are shown.

Let us illustrate with an example the previously described pattern of the structure of the IRs of $\hat{A}_1^{(1)}$. We denote by $(m_1, m_2)_{m_3}$ an IR of (Aff) algebra $A_1^{(1)}$, where the subscript label specifies the eigenvalue of $\Phi(H_3)$ which commutes with $\Phi(E_1)$ and $\Phi(F_1)$. In standard notation $m_1 = h, m_2 = k - h$ where k is the 'level' and h the HW of an IR of A_1. The second level of IR Λ_2 can be written as

$$(2,0)_{-1}\oplus(2,0)_0\oplus(0,2)_0\oplus(2,0)_1\oplus(0,2)_1\oplus(2,0)_2\oplus2(0,2)_2\oplus2(2,0)_3\oplus2(0,2)_3\oplus\cdots$$

the third level as

$$(2,1)_{-2}\oplus(0,3)_{-2}\oplus3(2,1)_{-1}\oplus2(0,3)_{-1}\oplus6(2,1)_0\oplus5(0,3)_0\oplus4(2,1)_1\oplus18(0,3)_1\oplus\cdots$$

The fact that IRs of different levels of Aff $A_1^{(1)}$ appear in the same IR of the algebra $\hat{A}_1^{(1)}$ suggests that a unified construction of IRs of any level of $A_1^{(1)}$ must be possible, in particular for the VO representation. This fact is not clear in the so called transverse VO realization while it is possible in the so called covariant VO construction [21]. Clearly this is a general feature for any Aff algebra.

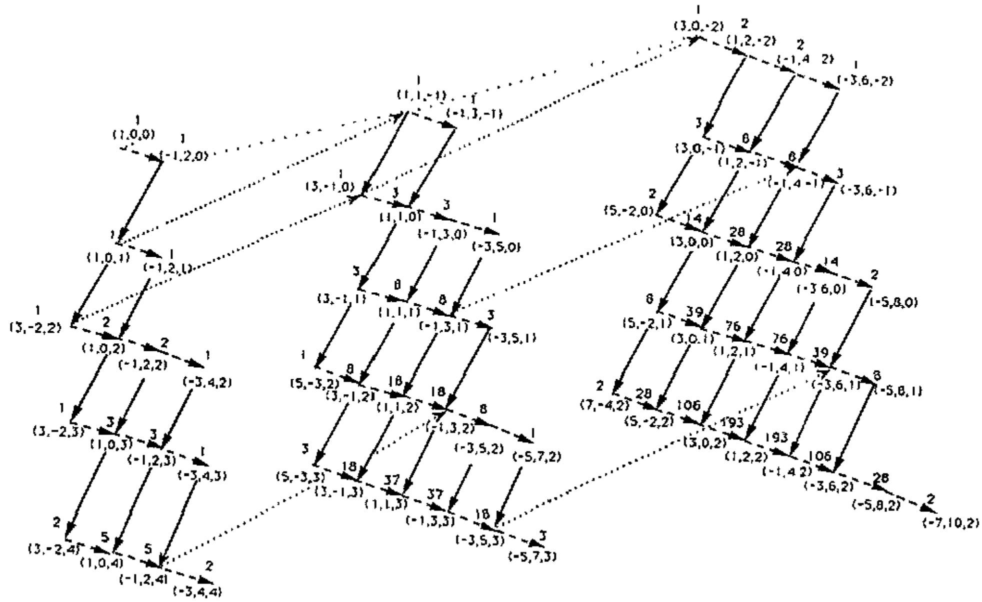


Figure 1. The first states of the first three levels of representation Λ_1 are reported. A state is denoted by a dot; the broken, full and dotted arrows denote the actions of $\Phi(F_1)$, $\Phi(F_2)$ and $\Phi(F_3)$, respectively. The triple of integers in brackets gives the values of m_i and the number on the dot gives the multiplicity of the state. A few spaced dotted lines without arrows are also drawn to identify some representations of the subalgebra A_2 better.

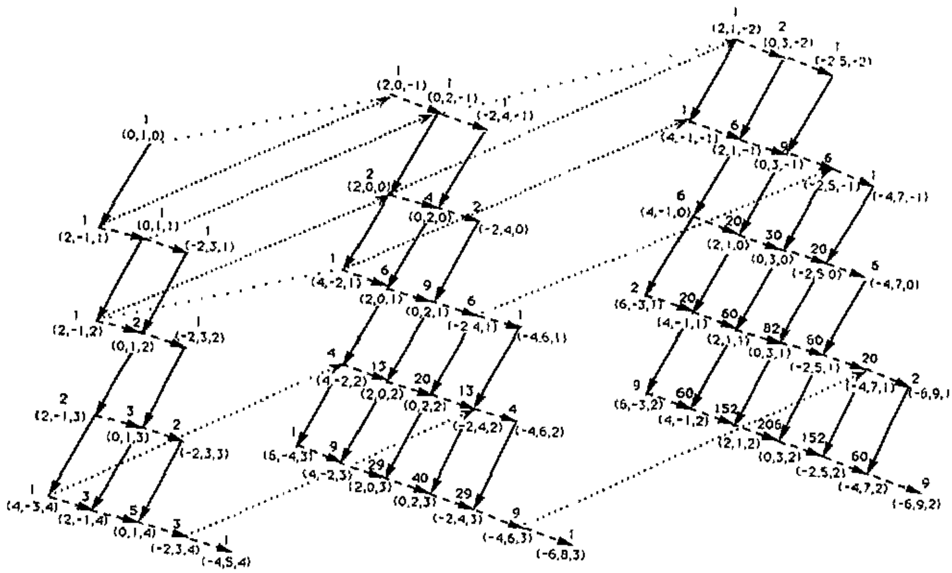


Figure 2. The first states of the first three levels of representation Λ_2 are shown.

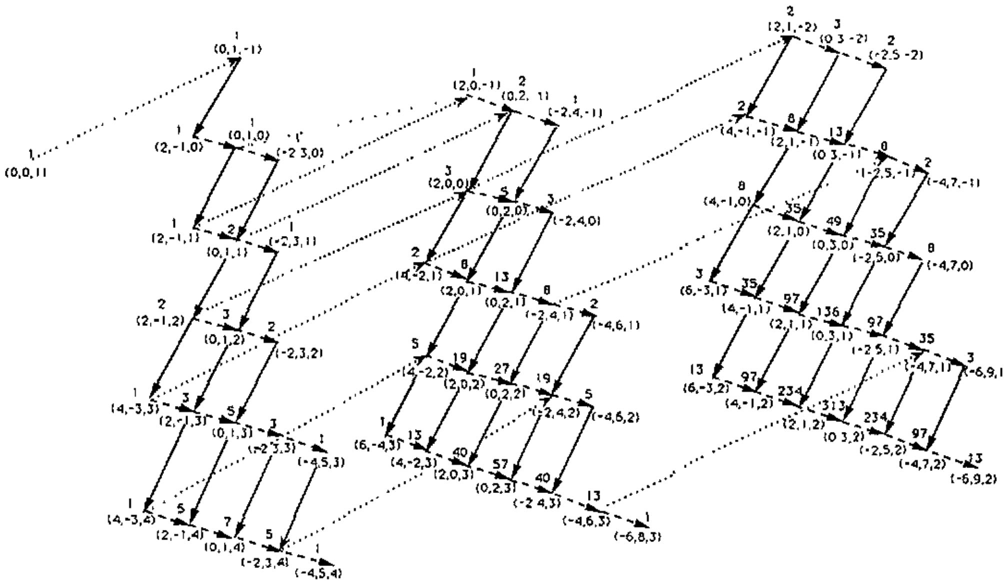


Figure 3. The first states of the first three levels of representation Λ_3 are shown.

5. Conclusions

The VO construction we have presented in section 3 is completely general and it applies to Hyp KMAs as well as to Lorentzian algebras. An important aspect, which has not been discussed in this paper, is connected with the conformal transformation properties of the fields which appear in the vertex operators. The discussion of the conformal behaviour of the field requires the construction of the Virasoro algebra associated with the KMA, which has been discussed, in full generality, by Borcherds in [15]. The conformal structure of the fields in the VO construction is extremely relevant for physical applications and it depends on the value of the dimension d. Another relevant aspect, which is not completely unrelated to the previous one, is the action of the GVOs on the Fock space of the representation. We will discuss both these aspects elsewhere [8]. The general structure of an IR for any Hyp KMA can be inferred from our discussion in section 4. However, many unsolved problems still remain. For instance, a general proof of the complete reducibility of a HW IR in terms of IRs of the affine subalgebra and a formula (at least formal) giving the decomposition of an IR of a Hyp KMA with respect to the affine subalgebra are missing.

Let us also remember that the string functions, which in the case of Aff KMAs allow the computation of the multiplicity of the weights, are not known for Hyp KMAs.

Acknowledgments

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Appendix

We give a proof of equation (69) which allows us to compute the multiplicity of weights by induction on the height $ht(\Lambda - \lambda)$.

In Kac's book [3] several formulae are given for the formal character (ch) of a HW module $L(\Lambda)$.

Let us start from equation

$$ch\, L(\Lambda) = \frac{\sum_{w\in W} \epsilon(w)e(w(\Lambda+\rho)}{\sum_{w\in W} \epsilon(w)e(w(\rho))} \tag{75}$$

where $\epsilon(w)$ is the parity of the Weyl reflection and $e(\lambda)$ is a formal exponential:

$$e(0) = 1 \qquad e(\lambda)e(\mu) = e(\lambda+\mu). \tag{76}$$

Multipling equation (75) by the denominator and by $e(-\rho)$, using the definition

$$ch\, L(\Lambda) = \sum_{\lambda \leqslant \Lambda} m_{L(\Lambda)}(\lambda)e(\lambda) \tag{76}$$

and a redefinition of the weight label λ of equation (77)

$$\lambda \to \lambda - w(\rho) + \rho. \tag{78}$$

Equation (75) can be rewritten as

$$\sum_{\lambda} e(\lambda) \sum_{w\in W} \epsilon(w) m_{L(\Lambda)}(\lambda + \rho - w(\rho)) = \sum_{w\in W} \epsilon(w)e(w(\Lambda+\rho) - \rho). \tag{79}$$

Multipling both sides of equation (79) by $e(\rho)$ using the invariance of the bilinear invariant form for Weyl transformations and the following properties (see proposition 11.4 of [3])

$$|\Lambda + \rho|^2 = |\lambda + \rho|^2 \qquad \text{iff } \lambda = \Lambda \tag{80}$$

we deduce equation (69) for $\lambda \neq \Lambda$.

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